

Moduli spaces for D-branes at the tip of a cone

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ABSTRACT: For physicists: We show that the quiver gauge theory derived from a Calabi-Yau cone via an exceptional collection of line bundles on the base has the original cone as a component of its classical moduli space. For mathematicians: We use data from the derived category of sheaves on a Fano surface to construct a quiver, and show that its moduli space of representations has a component which is isomorphic to the anticanonical cone over the surface.

KEYWORDS: Differential and Algebraic Geometry, D-branes, Topological Strings, AdS-CFT Correspondence.

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1. Introduction

Building on the mathematical work [1], there has recently been tremendous progress in the use of exceptional collections to derive quiver gauge theories for Calabi-Yau cones [2–7]. In this program, initiated in [8], one begins with a Fano Kähler-Einstein surface V . An exceptional collection on V is a collection of objects in the derived category of coherent sheaves on V that forms an analogue of a basis of a vector space. Given such a collection with nice enough properties, Bondal [1] constructs a quiver that has a derived category of representations equivalent to the derived category of coherent sheaves. It is shown in the above references how to ‘complete’ the quiver to obtain a new, more complicated quiver, whose category of representations corresponds not to V , but rather to the total space ω of the canonical bundle over V . In this paper we consider a related variety, $C(V)$, which (on the level of points) is obtained from ω by collapsing the zero section. It is so named because if the anticanonical bundle of V is very ample, then $C(V)$ is isomorphic to the cone over V in its anticanonical projective embedding. It follows from topological string theory [5] that the gauge theory on a D-brane located at the tip of the cone $C(V)$ is the quiver gauge theory corresponding to the completed quiver. Other Calabi-Yau cones can be obtained by undoing an orbifold as shown in [6]. Recently, Herzog and Karp [9] have shown how to find exceptional collections describing a large class of toric cones, and Verlinde and Wijnholt [10] have applied these techniques towards string phenomenology.

In string theory, we generally expect that if we have a D-brane probing a particular geometry, the moduli space of the gauge theory on the brane (or perhaps a particular branch thereof) should correspond in some way to the geometry being probed. A pioneering investigation of this question was in the work of Douglas, Greene and Morrison [11]. It is then an interesting question to ask about the moduli spaces of these quiver gauge theories derived from exceptional collections. In mathematical language, this is the moduli space of representations of the *completed* quiver. In this paper, we will address this question for exceptional collections consisting solely of line bundles. (This implies that the ranks of the

gauge groups are all one.) We will show that one component of the classical moduli space of the gauge theory is precisely the cone $C(V)$ on which the D-brane lives.

It is interesting to note that there exists a natural map from points on V (respectively ω) to the stack of isomorphism classes of representations of the original (respectively completed) quiver.¹ A representation consists of a vector space for each node in the quiver and a linear map for each arrow. In our case, the nodes of the quiver will correspond to the line bundles E_1, \dots, E_n in our exceptional collection, and the set of arrows from j to i will correspond to vectors in the vector space $\text{Hom}(E_i, E_j)$. Given a point p in our variety, there is a canonical quiver representation with vector spaces given by the fibers at p of the *dual* line bundles. Set theoretically, the moduli space of the gauge theory is in a quotient of a subset of the moduli stack. The compatibility of the stack theoretic and moduli theoretic approaches will be discussed in [12].

The main idea of our approach is as follows. It is well-known in gauge theory that the moduli space of vacua is parameterized by the gauge-invariant operators which for quiver gauge theories are all given by loops in the quiver. (In mathematical language, the moduli space is equal to the reduced variety underlying the affine GIT quotient of the space of representations by the complexified gauge group, which is defined as Spec of the ring of gauge-invariant functions.) A result of Bridgeland [3] tells us that the ring of based loops in our quiver, *i.e.*, loops through a given node, is isomorphic to the ring of sections of the anticanonical line bundle on V . It follows that the affine algebraic variety parameterized by the ring of based loops is isomorphic to the cone $C(V)$. There is still some difficulty arising from the mismatch between based loops and loops in general. This is why $C(V)$ turns out to be only one irreducible component (branch) of the moduli space.

In this discussion, we have set the Fayet-Iliopoulos terms to zero. (Mathematically, this means that we do not impose any stability condition on the space of representations.) Turning on these terms corresponds to (partially) desingularizing the tip of the cone². The relation of this to dibaryonic operators in the $SU(d)^n$ gauge theory is currently being pursued by the first author. We restrict in this paper to the situation where the cone is defined by the canonical line bundle. The extension to the undone orbifolds of [6] is straightforward. It is also an interesting question, in the mathematical context, to ask whether the uncompleted quiver has a stability condition such that its moduli space of stable representations has V as one of its components. This will be addressed in a future work [12].

This paper is organized as follows. In section 2, we briefly review the material that we need from the theory of exceptional collections in the derived category of an algebraic variety. In section 3, we describe quiver gauge theories and their moduli spaces and prove our main result. Finally, in section 4, we illustrate our result for the \mathbb{Z}_2 orbifold of the conifold.

2. Exceptional collections and quivers

In this section, we give an overview of the procedure for obtaining a quiver from an excep-

¹We thank David Ben-Zvi for calling our attention to this map.

²See [12, remark 3.4].

tional collection. For more details, see [1–4, 6]. An exceptional sheaf³ is a coherent sheaf, E , such that

$$\mathrm{Ext}^k(E, E) = \mathbb{C}^{\delta_{k0}}. \tag{2.1}$$

A strong exceptional collection is an ordered collection of exceptional sheaves, (E_1, \dots, E_n) , such that

$$i > j \Rightarrow \mathrm{Hom}(E_i, E_j) = 0, \tag{2.2}$$

$$\text{and } \forall i, j, \quad \mathrm{Ext}^k(E_i, E_j) = 0 \quad \forall k \neq 0. \tag{2.3}$$

Finally, the exceptional collection is called full if it generates the derived category. With this data, let

$$T = \bigoplus_{i=0}^n E_i, \tag{2.4}$$

and

$$A = [\mathrm{Hom}(T, T)]^{\mathrm{op}} = \bigoplus_{i \leq j} \mathrm{Hom}(E_i, E_j), \tag{2.5}$$

where the multiplication

$$\mathrm{Hom}(E_i, E_j) \times \mathrm{Hom}(E_k, E_l) \rightarrow \mathrm{Hom}(E_k, E_j) \tag{2.6}$$

is zero unless $i = l$ and is otherwise defined in the obvious manner. Bondal [1] constructs a quiver with relations whose nodes are indexed by the exceptional objects E_1, \dots, E_n , and whose path algebra is isomorphic to A . In other words, the vector space of paths from node j to node i modulo relations is canonically isomorphic to $\mathrm{Hom}(E_i, E_j)$, and the multiplication of Equation (2.6) corresponds to composition of paths. Furthermore, Bondal proves that the derived category of representations of this quiver is equivalent to the derived category of coherent sheaves on V .

Given such an exceptional collection, the doubly infinite collection of sheaves generated by the relation

$$E_i = E_{i+n} \otimes K_V, \quad i \in \mathbb{Z}, \tag{2.7}$$

is called a helix. Following Bridgeland [3], we call this helix simple if $\mathrm{Ext}^k(E_i, E_j) = 0$ for all $i \leq j \in \mathbb{Z}$ whenever $k \neq 0$. We next define the helix algebra,

$$\tilde{B} = \bigoplus_{k \geq 0} \prod_{j-i=k} \mathrm{Hom}(E_i, E_j), \tag{2.8}$$

where i and j run over the integers. This algebra has a natural \mathbb{Z} -action given by the isomorphism

$$\otimes K_V : \mathrm{Hom}(E_i, E_j) \longrightarrow \mathrm{Hom}(E_{i-n}, E_{j-n}). \tag{2.9}$$

The invariant subalgebra under this action is called the rolled-up helix algebra by Bridgeland; we will denote its opposite algebra by B . It has been long conjectured and finally

³This definition is most naturally applied to exceptional objects in the derived category, but in our case it is sufficient to work with honest sheaves.

shown in [7] that this algebra is the path algebra of a ‘completed quiver’ where the relations can be derived from a superpotential.

Recall that ω is defined to be the total space of the canonical bundle K_V on V , and let π denote the projection from ω to V . The nodes of the completed quiver correspond to the sheaves $\pi^*(E_i)$ on ω , and the arrows are defined such that the vector space spanned by the set of paths from node j to node i is isomorphic to

$$\text{Hom}(\pi^*(E_i), \pi^*(E_j)) = \bigoplus_{m \geq 0} \text{Hom}(E_i, E_j \otimes K_V^{-m}). \tag{2.10}$$

Note that these vector spaces are infinite dimensional, but each summand is finite dimensional. The grading by m in Equation (2.10) corresponds to the grading in the path algebra given by the number of times that a path circles the quiver. Since we will be assuming that the sheaves E_1, \dots, E_n are all line bundles, Equation (2.10) tells us that for all i , the algebra B_i of loops based at i is isomorphic to

$$\text{Hom}(\pi^*(E_i), \pi^*(E_i)) = \bigoplus_{m \geq 0} H^0(K_V^{-m}). \tag{2.11}$$

Even though the collection $\pi^*E_1, \dots, \pi^*E_n$ on ω is not exceptional, Bridgeland shows that the derived category of representations of B and, hence, of the completed quiver, is equivalent to the derived category of coherent sheaves on ω .

3. Moduli spaces of quiver gauge theories

Given a quiver with relations derived from a superpotential, the additional ingredient that we need to define a quiver gauge theory is a dimension vector. This is a vector of n integers, (d_1, \dots, d_n) , where n is the number of nodes in the quiver. The matter content then consists of a vector multiplet for each node associated with a $U(d_i)$ gauge group at that node and a chiral multiplet for each arrow that transforms in the fundamental of the head and the antifundamental of the tail. We will restrict our attention to the case in which the gauge groups are all $U(1)$. Thus each arrow corresponds to a complex number, and our configuration space is simply $\mathbb{C}^{\#\text{arrows}}$. We then impose the F-term and D-term constraints with all the FI-terms set to zero, and quotient by the gauge groups. The F-terms are the relations in the quiver as discussed in section 2, and they cut out a subvariety of $\mathbb{C}^{\#\text{arrows}}$. The D-term constraints are known in the mathematical literature as a moment map; imposing them and quotienting by the gauge group means taking a symplectic quotient. It is well known to both physicists and mathematicians [13, 14] that this corresponds to taking a GIT quotient by the complexified gauge group. With the FI-terms turned off, this means that we consider the affine variety $\text{Spec } R$, where R is the ring of gauge invariant functions on our subvariety. In fact, the relevant space for physics is not $\text{Spec } R$, but rather its underlying reduced variety, *i.e.*, the set of points cut out of affine space by the equations defining the ring R .

Let us now consider the total space ω of the canonical bundle on V . A polynomial function on ω may be thought of as an element of the section ring

$$S := \bigoplus_{m \geq 0} H^0(K_V^{-m}), \tag{3.1}$$

where m corresponds to the degree of the polynomial in the fiber direction. There is a natural projection from ω to $\text{Spec } S$, and it is an isomorphism away from the zero section, which gets collapsed to a point because V is projective. Let $C(V)$ be the underlying reduced variety of $\text{Spec } S$. If the anticanonical bundle is very ample then V is isomorphic to $\text{Proj } S$, and $C(V)$ is simply the cone over V in its anticanonical embedding. It is a theorem of LeBruyn and Procesi⁴ that the invariant ring of any quiver with any dimension vector is generated by the traces of automorphisms coming from loops in the quiver.⁵ Suppose that it is in fact generated by functions associated to loops that are based at a given node i . Then Equation (2.11) tells us that $R \cong B_i \cong S$, and therefore that $\text{Spec } R \cong \text{Spec } S$. Passing to the underlying reduced varieties, we conclude that the moduli space of vacua in the quiver gauge theory is isomorphic to $C(V)$.

Finally, we would like to eliminate the assumption that all invariants of the quiver are generated by loops based at a single node. In general, instead of an isomorphism between R and S , we have the following commutative diagram for each i , where φ_i is the isomorphism of Equation (2.11). The map Ψ is determined by the property that the above diagram

$$\begin{array}{ccc}
 B_i & \xleftarrow{\varphi_i} & S \\
 \sigma_i \searrow & & \nearrow \Psi \\
 & R &
 \end{array}$$

commutes for all i . The existence of such a map is guaranteed by the compatibility of the isomorphisms φ_i , and its uniqueness comes from the fact that the images of the various σ_i generate R as a ring. The injection $\sigma_i \circ \varphi_i$ induces a surjective map from the quiver moduli space $\text{Spec } R$ to $\text{Spec } S$, while Ψ induces a section of this map. We will prove below that there exists a nonzero element $\beta_i \in B_i$ such that σ_i becomes an isomorphism after inverting β_i . It follows that the inclusion induced by Ψ of $C(V)$ into the quiver moduli space is an isomorphism over the open set where β_i is nonzero. Since $C(V)$ is irreducible, this implies that the inclusion identifies $C(V)$ with an irreducible component of the moduli space.

Given any pair of nodes i and j , Equation (2.10) tells us that the space of paths from j to i can be identified with the vector space $\bigoplus_{m \geq 0} \text{Hom}(E_i, E_j \otimes K_V^{-m})$. Since K_V^{-1} is ample, this vector space is always nonzero. Choose a pair of nonzero paths p from i to j and q from j to i , and let β_i^j be the composition qp . If ℓ is a loop based at j , then

$$\sigma_i(\beta_i^j) \cdot \sigma_j(\ell) = \sigma_i(qp) \cdot \sigma_j(\ell) = \sigma_i(plq). \tag{3.2}$$

⁴See, for example, Lecture 10 of [15].

⁵It is interesting to note that if we replace the gauge groups by the special unitary groups $SU(d_i)$ we will have more invariants. In physics, these are called dibaryons, while in the mathematical literature, they are called semi-invariants of the quiver.

Put $\beta_i = \prod_{j \neq i} \beta_i^j$. Equation (3.2) tells us that $\sigma_i(\beta_i) \cdot \sigma_j(\ell) \in \sigma_i(B_i)$. Since the ring R is generated by functions associated to loops, this implies that $\sigma_i(\beta_i) \cdot R$ is contained in $\sigma_i(B_i)$. Thus, when we invert β_i , the inclusion σ_i becomes an isomorphism.

The precise mathematical theorem that we have proven may be stated as follows.

Theorem 1. *Let E_1, \dots, E_n be a full, strong, exceptional collection of line bundles on a Fano surface V , generating a simple helix. Then $C(V)$ includes into the moduli space of S -equivalence classes⁶ of representations of the associated ‘completed quiver’ with dimension vector $(1, \dots, 1)$, and the image is the canonical reduced subscheme of an irreducible component.*

4. An example

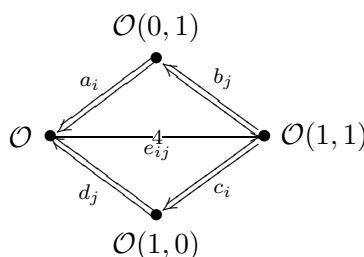


Figure 1: The completed quiver for $\mathbb{P}^1 \times \mathbb{P}^1$.

To illustrate our theorem, we consider an example in which $V \cong \mathbb{P}^1 \times \mathbb{P}^1$, and $C(V)$ is the \mathbb{Z}_2 orbifold of the famous conifold. The line bundles $\mathcal{O}, \mathcal{O}(0, 1), \mathcal{O}(1, 0)$, and $\mathcal{O}(1, 1)$ form a full strong exceptional collection; its quiver is shown in Figure 1, with i, j, k and l running from 1 to 2. The 4 in the middle of the central arrow means that there are four arrows from \mathcal{O} to $\mathcal{O}(1, 1)$. Let $\{x_1, x_2\}$ be a basis of sections of $\mathcal{O}(0, 1)$, and $\{y_1, y_2\}$ a basis of sections of $\mathcal{O}(1, 0)$. The arrows a_i and c_i correspond to multiplication by x_i , and b_j and d_j correspond to multiplication by y_j . This gives the relations

$$a_i b_j = d_j c_i \quad \text{for all } i, j. \tag{4.1}$$

In addition, the arrows e_{ij} correspond to multiplication by $x_i y_j$ leading to further relations. These relations can all be derived from the superpotential

$$W = (a_1 b_1 - d_1 c_1) e_{22} - (a_1 b_2 - d_2 c_1) e_{21} - (a_2 b_1 - d_1 c_2) e_{12} + (a_2 b_2 - d_2 c_2) e_{11}. \tag{4.2}$$

Let us consider only those loops that are based at the node $\mathcal{O}(0, 1)$. There are naïvely $2 \times 4 \times 2 = 16$ loops through this node that cycle exactly once around the quiver. Modulo relations, however, we find that the ring of based loops is generated by nine monomials and may in fact be identified with the subring of $\mathbb{C}[x_1, x_2, y_1, y_2]$ spanned by all monomials with the same even degree in both x and y . One can see by explicit calculation that this gives

⁶This means that we identify two isomorphism classes if their closures intersect in the moduli stack.

the \mathbb{Z}_2 orbifold of the conifold, but it is also possible to see this more geometrically. This ring is precisely the homogeneous coordinate ring of $\mathbb{P}^1 \times \mathbb{P}^1$ in its projective embedding defined by composing the 2-uple embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^2 \times \mathbb{P}^2$ with the Segré embedding of $\mathbb{P}^2 \times \mathbb{P}^2$ in \mathbb{P}^8 . The hyperplane bundle on \mathbb{P}^8 pulls back to $\mathcal{O}(2, 2) = K_V^{-1}$ over $\mathbb{P}^1 \times \mathbb{P}^1$, so we immediately see that our variety is the affine cone we are looking for. In this case, the invariant ring is an integral domain, and, consequently, any loop that does not pass through the node $\mathcal{O}(0, 1)$ is equivalent in the path algebra to a linear combination of loops that do.

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